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K-theoretical boundary rings in $\mathcal{N} = 2$ coset models

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Abstract

A boundary ring for $\mathcal{N} = 2$ coset conformal field theories is defined in terms of a twisted equivariant K-theory. The twisted equivariant K-theories ${}^{\tau}K_H(G)$ for compact Lie groups (G, H) such that G/H is hermitian symmetric are computed. These turn out to have the same ranks as the $\mathcal{N} = 2$ chiral rings of the associated coset conformal field theories, however the product structure differs from that on chiral primaries. In view of the K-theory classification of D-brane charges this suggests an interpretation of the twisted K-theory as a 'boundary ring'. Complementing this, the $\mathcal{N} = 2$ chiral ring is studied in view of the isomorphism between the Verlinde algebra $V_k(G)$ and ${}^{\tau}K_G(G)$ as proven by Freed, Hopkins and Teleman. As a spin-off, we provide explicit formulae for the ranks of the Verlinde algebras.

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1. Introduction

Twisted K-theory has recently attracted much attention in various areas in string theory and conformal field theory. The two main applications that have crystalized so far are the classification of D-brane charges in backgrounds with non-trivial NSNS 3-form flux, as well as the beautiful result by Freed, Hopkins and Teleman (FHT) [1,2,3], which identifies the Verlinde algebra for a compact Lie group G with the twisted equivariant K-theory ${}^{\tau}K_G(G)$. The main motivation for the present work was to set out and ask, whether there is a similar relation between twisted equivariant K-theory and the $\mathcal{N}=2$ chiral ring for coset conformal field theories. For coset models with groups (G,H) and common centre Z, the relevant twisted equivariant K-theories are ${}^{\tau}K_{H/Z}(G)$. We compute these K-theories in general, thus extending the results of [4], and show them to be of the same rank as the $\mathcal{N}=2$ chiral ring of the coset model in question. However, interestingly, the product defined on the K-theory differs from the one on chiral primaries. This new ring-structure will be referred to as the *(K-theoretical) boundary ring*, as it has a natural interpretation in terms of a product structure on (classes of) D-branes. This may have a close correspondence with the algebra of BPS states defined in [5,6]. A CFT-discussion of this product will appear in [7], extending some of [8].

In the context of D-branes in supersymmetric WZW and coset theories the identification of D-brane charges with twisted K-theory has been established in various instances. The case of WZW models and the corresponding computations of twisted K-theories for compact Lie groups is discussed in [9,10,11,12,13,14,15,16,17,18,19]. For some of the $\mathcal{N}=2$ coset models, namely the Grassmannian cosets, the charge lattices for the D-branes were obtained in [20,8,21,22,23,24] and the relevant twisted, equivariant K-theories have been computed in [4]. One task, which will be accomplished in the present paper is to generalize the computation in [4] to all $\mathcal{N}=2$ Kazama-Suzuki coset models [25]. Despite the successful description of D-brane charges in these theories by means of twisted K-theory, a conceptual understanding of this relation still needs to be elucidated. Some progress to this end has been obtained for topological theories in [26].

On the other hand, the theorem by FHT [1,2,3] provides a concrete correspondence between conformal field theoretical data, such as the Verlinde fusion ring, and topology. From a CFT (or rather TFT) point of view, the result by FHT on ${}^{\tau}K_G(G)$ can be interpreted as a statement about the D-brane charges in the G/G gauged WZW model, which is in fact topological. The next simplest such theories are the Kazama-Suzuki models –

which are conformal, but have $\mathcal{N} = 2$ worldsheet supersymmetry, and thus would allow for a topological twisting. The present paper provides a discussion of these $\mathcal{N} = 2$ coset models in light of the results in [1,2,3]. In summary we shall prove the following

Theorem. *Let G be a simple, simply-connected, connected Lie group and H a connected, maximal rank subgroup of G, such that G/H is hermitian symmetric. Let Z be the common centre of G and H, which is assumed to act without fixed points, and denote by $\mathcal{R}_{cp}^{(G,H)}$ the chiral ring of the corresponding $\mathcal{N} = 2$ coset conformal field theory. Then*

$$\mathit{rank}\left({}^{\tau}K_{H/Z}^{\dim(G)}(G) \right) \cong \mathit{rank}\left(\mathcal{R}_{cp}^{(G,H)} \right) , \tag{1.1}$$

and the ring structure on the K-theory is

$${}^{\tau}K_{H/Z}^{\dim(G)}(G) \cong \left(\frac{R_H}{I_k(G)} \right)^Z . \tag{1.2}$$

Here, ${}^{\tau}K_{H/Z}^{\dim(G)}(G)$ is the twisted H/Z-equivariant K-theory of G, where the action of H on G is by conjugation, R_H denotes the H-representation ring, $I_k(G)$ is the Verlinde ideal of G and the Z-invariant part is taken on the RHS. The twisting $\tau \in H_H^3(G)$ is related to the level of the coset model by $\tau = \kappa[\mathbf{H}]$, where $[\mathbf{H}]$ is the generator of $H_H^3(G)$ and $\kappa = k + g^\vee$.

Assuming the K-theory classification of D-brane charges [27,28], a straight forward implication of the theorem is the following

Corollary. *The charge lattice for D-branes in the Kazama-Suzuki coset models associated to (G, H) is of the same rank as the $\mathcal{N} = 2$ chiral ring.*

As emphasized, the ring structure on the K-theory is however somewhat different from the one on the chiral ring, thus motivating the

Definition. *The (K-theoretical) boundary ring $\mathcal{B}_k^{(G,H)}$ of the $\mathcal{N} = 2$ coset model is defined as the ring in (1.2).*

The plan of this paper is as follows. In section 2, we present our main result by computing the twisted equivariant K-theories relevant for all Kazama-Suzuki (KS) cosets, generalizing [4], and prove that the ranks of the K-theory agrees in all instances with that of the chiral ring. We provide the explicit formulae for the ranks, including the ranks of the Verlinde algebras (which to our knowledge have not been explicitly documented in the literature), the computation of which we provide in appendix A. A new boundary ring, motivated by K-theory, is defined in section 3, and its relation to the standard (bulk) chiral ring is discussed. The construction of elements of the chiral ring as K-theory classes using families of affine Dirac operators is provided in section 4 and we close in section 5 with discussions and outlook.

2. Twisted equivariant K-theory for Kazama-Suzuki models

GKO coset models [29] (see also [30] for further references) with $\mathcal{N}=1$ supersymmetry associated to a compact Lie group G and a maximal rank subgroup H, with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, have chiral algebra

$$\mathcal{A} = \frac{\hat{\mathfrak{g}}_k \oplus \widehat{\mathfrak{so}}(\dim(\mathfrak{g}/\mathfrak{h}))_1}{\hat{\mathfrak{h}}_{k+h^\vee_{\mathfrak{g}}-h^\vee_{\mathfrak{h}}}}\,, \tag{2.1}$$

and are known to be $\mathcal{N}=2$ supersymmetric if the right coset space G/H is a hermitian symmetric space [25]. The so-obtained Kazama-Suzuki (KS) coset models are thus classified by the (irreducible) hermitian symmetric spaces, which fall into the following classes [31,25]:

G	H
$SU(n+m)$	$SU(n)\times SU(m)\times U(1)$
$SO(n+2)$	$SO(n)\times SO(2)$
$SO(2n)$	$SU(n)\times U(1)$
$Sp(2n)$	$SU(n)\times U(1)$
E_6	$SO(10)\times U(1)$
E_7	$E_6\times U(1)$

Table 1: *Hermitian Symmetric Spaces.*

Recall, that G/H is hermitian symmetric iff the following condition is satisfied on the Lie algebras: consider the orthogonal decomposition $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{p}$, with $[\mathfrak{h},\mathfrak{h}]\subset\mathfrak{h}$, then the condition reads

$$[\mathfrak{h},\mathfrak{p}]\subset\mathfrak{p}\,,\qquad [\mathfrak{p},\mathfrak{p}]=0\,, \tag{2.2}$$

i.e., in particular $\mathfrak{p}$ is abelian. More generally the coset is Kähler if $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{p}$. Note, that under these circumstances, the maximal tori of G and H can be chosen to coincide.

In the following we shall be interested in the corresponding twisted equivariant K-theories ${}^\tau K_H(G)$. The K-theories in the case of projective cosets $SU(n+1)/U(n)$ and

generalized superparafermions $SU(n+1)/U(1)^n$ have been computed in [4]. The computation there relied on the group G being a connected, simply-connected Lie group. The first step in order to generalize these K-theory computations to all the coset models corresponding to the spaces listed in table 1, is to note that for hermitian symmetric spaces G/H [31](Theorem 4.6)

$$G/H \cong \widetilde{G}/\widetilde{H}\,, \tag{2.3}$$

where $\widetilde{G}$ denotes the covering group of G. In particular, the cosets based on $SO(n)$ can be replaced by the corresponding $Spin(n)$ cosets, and thus (2.3) allows to treat the K-theory computation uniformly for all KS models, assuming that all the groups are simply-connected.

In the following we shall always assume that G is a simple, simply-connected, connected Lie group, and H a connected, maximal rank subgroup of G. Under these circumstances, we have shown in [4] that the twisted equivariant K-theories can be computed using the observation that

$$^{\tau}K_H(G) = {}^{\tau}K_G(G \times_H G^L)\,, \tag{2.4}$$

where G^L is acted upon by left-multiplication, whereas the action on the remaining groups is by conjugation. This yields by the equivariant Künneth theorem

$$^{\tau}K_H(G) = {}^{\tau}K_G(G) \otimes_{R_G} R_H = \frac{R_G}{I_k(G)} \otimes_{R_G} R_H\,, \tag{2.5}$$

where we invoked the result of Freed, Hopkins and Teleman [2,3]

$$^{\tau}K_G(G) = V_k(G) = \frac{R_G}{I_k(G)}\,. \tag{2.6}$$

V_k denotes the Verlinde algebra♮ and $I_k(G)$ the Verlinde ideal of G at level k, which is specified by the twisting τ. H acts upon G by the conjugation action. Note that (2.6) is in fact an algebra isomorphism, where the product on V_k is the fusion product and on the twisted K-theory side it is the Pontryagin product [1,2,3]. We shall discuss the induced

♮ As coefficients in $\mathbb{Z}$ are used, the correct terminology is in fact *ring* instead of *algebra*. However, we shall continue to refer to this as the *Verlinde algebra*, making the coefficient ring/field explicit, when necessary.

product structure on ${}^\tau K_H(G)$ in the next section. Further, since H is connected and of maximal rank, R_H is free as an R_G-module [32], so that

$$ {}^\tau K_H(G) = \frac{R_H}{I_k(G)} \,. \tag{2.7} $$

Thus, in order to determine the rank of the K-theory, we need to compute the rank of R_H as an R_G-module, as well as the rank of the Verlinde algebra. In order to acquire the former, we recall that by [32]

$$ K(G/H) = R_H \otimes_{R_G} \mathbb{Z} \,, \tag{2.8} $$

i.e., the rank of R_H can be computed via the untwisted K-theory of the symmetric spaces (left-action cosets) in question. By a theorem of Atiyah and Hirzebruch [33] (Theorem 3.6)

$$ K(G/H) = \mathbb{Z}^{\frac{|W_G|}{|W_H|}} \,, \tag{2.9} $$

where W_G denotes the Weyl group of G (a more detailed discussion of these K-groups can be found in [34]). Thus we arrive at the final result

$$ {}^\tau K_H(G) = \mathbb{Z}^{d_k(G)\frac{|W_G|}{|W_H|}} \,, \tag{2.10} $$

where

$$ d_k(G) = \mathrm{rank}(V_k(G)) \,. \tag{2.11} $$

In order to acquire a totally explicit expression for the rank, we need to determine $d_k(G)$ in each of the above cases. One can *e.g.* determine $d_k(G)$ combinatorially. $d_k(G)$ is equal to the number of integrable highest weights at level k, *i.e.*, it can be determined as the number of solutions to the inequality

$$ (\Lambda, \theta) \leq k \,, \tag{2.12} $$

where Λ denotes the highest weight, and θ the highest root. That is, one has to count the non-negative integer solutions for the Dynkin labels $\{\Lambda^{(i)}\}$ respecting the inequality

$$ \sum_{i=1}^{n} \Lambda^{(i)} a_i^\vee \leq k \,, \tag{2.13} $$

where $a_i^\vee$ are the dual Coxeter labels. Doing the combinatorics, the details of which we provide in appendix A, implies table 2 in appendix A.

Due to the non-trivial selection rules in the KS coset theories, the relevant K-groups that should classify the D-brane charges are in fact ${}^{\tau}K_{H/Z}(G)$, where Z is the common centre of G and H. We shall restrict our attention to the cases, when Z acts without fixed points. As explained in [4], this reduces the rank of the K-theory by a factor equal to the lengths, $l(Z)$, of the orbits of Z acting on the geometric invariant theoretical (GIT) quotient $H//H$, so that

$$ {}^{\tau}K_{H/Z}(G) = \mathbb{Z}^{\frac{d_k(G)}{l(Z)} \frac{|W_G|}{|W_H|}} . \tag{2.14} $$

Note that the charge lattice is thus precisely of the same rank as the chiral ring of the KS models as determined in [35].

3. $\mathcal{N} = 2$ boundary rings from K-theory

In view of the result (2.14) it is very tempting to conjecture that the chiral ring of an $\mathcal{N} = 2$ coset model is given by a twisted, equivariant K-theory – much like the Verlinde algebra is ${}^{\tau}K_G(G)$. In this section we will discuss this correspondence in some detail, and arrive at the conclusion that the twisted K-theory defines a ring, whose underlying $\mathbb{Z}$-module structure is the same as the chiral ring (*i.e.*, they have the same ranks), but the product structure is different.

3.1. Proposal for an $\mathcal{N} = 2$ boundary ring

The K-theory for the KS-coset models naturally comes equipped with a product structure. Tracing this through our computations in the last section, we see that this is the induced ring structure from ${}^{\tau}K_G(G)$, which by FHT is the Pontryagin product on K-theory classes[♭] and agrees with the fusion product in the Verlinde algebra of G.

Let us assume the validity of the conjectural one-to-one correspondence between K-theory classes and classes of D-branes (where the equivalence is say with respect to boundary RG flows). Put into this context, our K-theory computation suggests to define the following (K-theoretical) boundary ring

$$ \mathcal{B}_k^{(G,H)} := {}^{\tau}K_{H/Z}^{\dim(G)}(G) \cong \left(V_k(G) \otimes_{R_G} R_H\right)^Z , \tag{3.1} $$

[♭] Note that this makes use of the product on G in an essential way.

where the Z-invariant part is taken on the RHS. We shall mostly abbreviate this as $\mathcal{B}_k$. Let us stress, that this is different from the chiral ring of the coset model. In particular, (3.1) can be written as a quotient of the H-representation ring by the Verlinde ideal of G

$$\mathcal{B}_k \cong \left(\frac{R_H}{I_k(G)} \right)^Z \ . \tag{3.2}$$

The D-brane interpretation of this is twofold: firstly, the K-theory charge lattice seems to be spanned already by the Cardy branes (labeled by chiral primaries). This is presumably due to the worldsheet $\mathcal{N} = 2$ supersymmetry. An interesting exercise, which might elucidate this point is to analyze the charges in the topologically twisted Kazama-Suzuki models. The second point is, that the K-theory comes naturally with a product structure, which therefore corresponds to a product on equivalence classes of D-branes. Having said this, the ring structure should then in particular account for the charge relations, that can be derived *e.g.* from a worldsheet point of view. A complementary CFT discussion of this matter will appear in [7]. For the $SU(2)/U(1)$ coset model, the next section will illustrate that the ring structure does indeed respect the charge relations derived in [20].

3.2. $\mathcal{N} = 2$ super-minimal models

The simplest KS models are the super-minimal models/superparafermions, realized in terms of $SU(2)/U(1)$. Recall that

$$\begin{aligned} {}^{\tau}K_{U(1)}(SU(2)) &= \frac{R_{u(1)}}{I_k(SU(2))} = \frac{\mathbb{Z}[\zeta, \zeta^{-1}]}{\langle \mathrm{Sym}_{k+1}(\zeta + \zeta^{-1}) = 0 \rangle} \\ &= \big\langle 1, \zeta, \zeta^{-1}, \cdots ; \ \mathrm{Sym}_{k+1}(\zeta + \zeta^{-1}) = 0 \big\rangle \ . \end{aligned} \tag{3.3}$$

Here, $\mathrm{Sym}_n(x)$ denotes the symmetric polynomial of degree n in x and the generator, Λ, of $R_{SU(2)}$ has been decomposed with respect to $U(1)$, *i.e.*, $\Lambda = (\zeta + \zeta^{-1})$. For the boundary ring one needs to consider ${}^{\tau}K_{U(1)/Z}(SU(2))$, *i.e.* take the invariant part under the common centre $Z = \pm 1$, which acts on the representations as $\zeta \mapsto -\zeta$, thus removing the odd powers of ζ. Hence

$$\mathcal{B}_k = \big\langle 1, \zeta^2, \zeta^{-2}, \cdots ; \ \mathrm{Sym}_{k+1}(\zeta + \zeta^{-1}) = 0 \big\rangle \ . \tag{3.4}$$

In particular, the rank is $k + 1$ and does indeed agree with the one of the chiral ring. However, the relations in the latter are $\Lambda^{k+1} = 0$, whereas they are $\mathrm{Sym}_{k+1}(\zeta + \zeta^{-1}) = 0$

in the K-theory, so that the ring structures differ[♯]. For instance at $k = 1$ the relation reads $\zeta^2 + 1 + \zeta^{-2} = 0$, which does not factor within the ring $\mathcal{B}_k$.

Note that if one considers only the homogeneous part in Λ of the Verlinde ideal, that is in this case the ideal $J = \langle (\zeta + \zeta^{-1})^{k+1} = 0 \rangle$, the resulting ring would agree with the chiral ring. *E.g.* for $k = 1$, the relation is $\zeta^2 + 2 + \zeta^{-2} = 0$, which generates the same ideal as $\zeta^4 + 2\zeta^2 + 1 = (\zeta^2 + 1)^2 = 0$. Thus, setting $x = \zeta^2 + 1$, we obtain the same relation as in the chiral ring. Let us stress that this is however *not* what one obtains from the K-theory and thus in $\mathcal{B}_k$. For level 1, the latter is the quantum deformation of the chiral ring (which is simply $H^*(\mathbb{C}P^1)$) with the deformation parameter set to -1. This observation has in fact been made in [8] for the ring obtained from boundary intersection matrices of D-branes in KS models.

Note also, that the boundary ring nicely encodes the charge relations in the $\mathcal{N} = 2$ super-minimal models. Geometrically the (A-)branes are lines in the disc target space [20]. The shortest lines correspond to the basis of the charge lattice, with the relation that the closed ring of shortest branes is trivial [20,36]. This is precisely the relation in $\mathcal{B}_k$, under the identification of the short branes with the generators ζ^l, $l \in 2\mathbb{N}$.

Remark

We should digress, and make a remark upon the relation of our results for the super-minimal models to the recent computations in [37]. The computation of D-brane charges in the $\mathcal{N} = 2$ minimal models $su(2)_k/u(1)$ in the paper in question yielded B-brane charges $\mathbb{Z}_{k+2}$ and as well as A-brane charges $\mathbb{Z}^{k+1}$. This is not in contradiction with the present results and the ones in [4], as the computation in the latter is for the diagonal modular invariant, whereas the computation in [37] seems to be for a $(-1)^F$ orbifold thereof (see also comments on this matter in [38]). A detailed discussion of this point will appear in a forthcoming paper [39]. In brief, for modular invariants, which are obtained as simple-current extensions of the diagonal modular invariant, one has to incorporate the additional equivariance with respect to the simple-current in the corresponding K-theory computation. For non-trivial actions on the fermions (*i.e.*, on the $\mathfrak{so}(2d)_1$ factor), one has additional twist choices apart from $H^3(X)$, and for instance the Hopkins K-groups $K_\pm$ [40] (see also [28,41]) are relevant. An example of this has been worked out in [15].

[♯] I thank S. Fredenhagen for discussions on this point.

3.3. Boundary ring versus chiral ring: Level 1 discussion

In [35,42] a geometrical interpretation of the chiral ring has been given for level 1 KS models, based on simply-laced groups. There it was proven that

$$\mathcal{R}^{(G,H)}_{k=1} \cong H^*(G/H)\,, \tag{3.5}$$

where the RHS is the cohomology ring for the right-action coset space. Let's see what our proposal yields in this instance. Our twisted K-theory computation results in

$${}^{\tau}K_H(G) \cong V_{k=1}(G) \otimes_{R_G} K(G/H)\,. \tag{3.6}$$

Further note that for simply-laced groups, $|Z| = |V_{k=1}(G)|$. Taking the Z-equivariance into account, we infer that

$$\mathrm{rank}(\mathcal{R}^{(G,H)}_{k=1}) = \ \mathrm{rank}({}^{\tau}K_{H/Z}(G))\,. \tag{3.7}$$

However the product structure on the chiral ring, which in this case is the wedge product on the cohomology ring of G/H, differs from the one on

$${}^{\tau}K_H(G) \cong \left(\frac{R_H}{I_1(G)}\right)^Z\,. \tag{3.8}$$

The reason is again that Z does not have a homogeneous action on the generators of $V_1(G)$. Again, one sees that taking just the highest degree term in the relations for the Verlinde ideal at level 1 would give rise to the chiral ring.

3.4. Boundary ring versus chiral ring: Relation to fusion rings

It was observed in [35,42], that the chiral ring of an $\mathcal{N}=2$ coset model can be related to Verlinde algebras (*i.e.*, fusion rings). This is most concisely explained by Witten in [43]. The chiral ring is obtained by quantizing the following phase space

$$P_R = \frac{T \times T/Z}{W_H}\,, \tag{3.9}$$

where T is the maximal torus of G (and so also of H).

On the other hand one can relate the chiral ring to the representation ring of H by noting that [44] the Verlinde algebra for H is obtained by quantizing the space

$$P_V = \frac{T \times T}{W_H}\,. \tag{3.10}$$

This yields the key relation, that after quantization we obtain

$$\mathcal{R}_{cp}^{(G,H)} = \big(V_{\tilde{k}}(H)\big)^{Z} \ , \tag{3.11}$$

so that the chiral ring of the coset is (in fact only as a lattice) isomorphic to the Z-invariant part of a quotient of R_H by an ideal, which is not the Verlinde ideal of G. However, this line of argument is to be taken with a grain of salt, as (3.11) only seems to hold as a lattice isomorphism, one cannot infer straight away that the product on the RHS of (3.11) is different from the chiral ring product, *e.g.* by considering simple examples.

3.5. Product structure

Let us briefly discuss the product on the K-theory, without making use of the relation to R_H. One can define the Pontryagin product on the K-theory classes, yet again, in complete analogy to [1], namely

$$m_{pon} \ : \qquad {}^{\tau}K_{H/Z}(G) \otimes {}^{\tau}K_{H/Z}(G) \ \rightarrow \ {}^{\tau}K_{H/Z}(G)\,. \tag{3.12}$$

To establish this, consider the multiplication on the group $m : \, G \times G \rightarrow G$ and assume that the twisting respects this, in the sense that $m^*(\tau)$ factorizes over the two groups. Then by pushing forward along m induces the product

$${}^{\tau}K_{H/Z}(G) \, \otimes_{R_G} \, {}^{\tau}K_{H/Z}(G) \ \rightarrow \ {}^{\tau\oplus\tau}K_{H/Z\times H/Z}(G\times G) \ \rightarrow \ {}^{\tau}K_{H/Z}(G\times G)\,, \tag{3.13}$$

where the first map is application of the Künneth theorem. This is equivalent to the product, that we encountered in section 2, which we obtained by invoking the product on ${}^{\tau}K_G(G)$ of FHT, *i.e.*, the product on (3.1). On the other hand the product on the chiral ring represented in terms of coset fields is the fusion product induced from the Verlinde algebras of G and H. More precisely, it is the fusion product on pairs of primaries in $V_k(G)$ and $V_{k+g^\vee}(H)$, respectively, modulo selection and identification rules. In particular this is distinct from (3.1).

4. The $\mathcal{N}=2$ chiral ring and twisted equivariant K-theory

Recall that the FHT theorem states the isomorphism

$$ {}^{\tau}K_{G}^{\dim(G)}(G) \cong V_k(G)\,, \tag{4.1} $$

where $\tau = k + h^{\vee}$ is the twist-class in $H^3_G(G)$. The proof of this theorem in [3] is based on constructing the K-theory classes from families of affine Dirac operators on loop space. In this section we divert from the main thread of the paper and explain how a related construction can be put forward for the chiral ring. Only at the end, we shall comment again on the relation to the boundary ring.

As a motivation, first recall the construction of the chiral ring in [35] for the $\mathcal{N}=2$ coset theory with chiral algebra (2.1). The essential ingredient is an index computation for the $\mathcal{N}=2$ supercharges (affine Dirac operators), more precisely, the chiral primaries (or, by spectral flow, the RR ground states) are solutions to

$$ G^{\pm}|\,\varphi\,\rangle = 0\,, \tag{4.2} $$

where the affine Dirac operators are defined as (neglecting for the present discussion irrelevant prefactors)

$$ G^{\pm} = \not{D}^{\pm}_{L\hat{\mathfrak{g}}/L\hat{\mathfrak{h}}} = \left(J^{\mp\alpha}_{-n}\psi^{\pm\alpha}_{n} - \frac{1}{12}\, f^{\pm\gamma}_{\pm\alpha\,\pm\beta}\,\psi^{\pm\alpha}_{n}\,\psi^{\pm\beta}_{m}\,\psi^{\mp\gamma}_{-n-m} \right)\,, \tag{4.3} $$

where ψ^{α} are adjoint fermions transforming under the $so(2d)_1$ algebra, and the α's are summed over the positive roots of $\mathfrak{g}/\mathfrak{h}$. Note that for trivial $\mathfrak{h}$, $\not{D}^{+}$ is precisely the affine Dirac operator that enters the construction of the G/G twisted K-theory in [3]. In fact, the zeroes of the affine Dirac operator $\not{D}^{+}$ for both the WZW and coset models have been computed by Landweber in [45], in analogy to the analysis for finite-dimensional Lie groups by Kostant [46].

The key idea of Mickelsson [47] and Freed-Hopkins-Teleman [3], which provides the link to twisted K-theory, is to couple the Dirac-operator on $L\mathfrak{g}$ to an $L\mathfrak{g}^{*}$-valued gauge field. The kernel of the affine Dirac operator on $L\mathfrak{g}$ is trivial, however the gauge-coupled Dirac operator (in the following referred to as the FHT Dirac family) has non-trivial kernel, to which one can associate a twisted K-theory class on G.

4.1. Review of Dirac-family construction for the Verlinde algebra

First we review the Dirac-family construction for (4.1). Throughout this section we shall adopt the conventions of [3]. Consider the following (gauge-coupled) family of affine Dirac operators

$$\not{D}_\mu^{FHT} = \not{D}_{L\mathfrak{g}} + i\psi(\mu)\,, \tag{4.4}$$

for $\mu \in L\mathfrak{g}^*$ (*cf.* (4.11) in [47], (11.2) in [3]), where the affine Dirac operator on $L\mathfrak{g}$ is defined by

$$\not{D}_{L\mathfrak{g}} = \sum_{\alpha\in\Delta_{L\mathfrak{g}}} \left(J^\alpha_{-n}\psi^{-\alpha}_n - \frac{1}{12}\sum_{\beta,\gamma\in\Delta_{L\mathfrak{g}}} f^\gamma_{\alpha\,\beta}\,\psi^\alpha_n\,\psi^\beta_m\,\psi^{-\gamma}_{-n-m}\right)\,. \tag{4.5}$$

Furthermore, $\psi(\mu)$ denotes Clifford multiplication with this element of Cliff($L\mathfrak{g}$). $\not{D}$ acts on $\mathcal{H}_\lambda\otimes\mathcal{S}_{L\mathfrak{g}}$, where $\mathcal{H}_\lambda$ is an integrable highest weight representation of $\widehat{L\mathfrak{g}}$, and $\mathcal{S}$ the spin representation. The key properties of this family are: The Dirac family (4.5) is equivariant under the co-adjoint action of LG. This can be mapped to the gauge action of LG on the space $\mathfrak{A}$ of $\mathfrak{g}$-valued connections on the circle by identifying the level k hyperplanes in $\widehat{L\mathfrak{g}}^*_k$ in the following fashion

$$\begin{aligned} k\Lambda + L\mathfrak{g}^* &\to \mathfrak{A} \\ k\Lambda + \mu &\mapsto \frac{d}{dt} + \frac{\mu}{k}\,. \end{aligned} \tag{4.6}$$

Further, acting on a highest weight module $\mathcal{H}_\lambda\otimes\mathcal{S}$, the kernel of $\not{D}_\mu$ is localized on the co-adjoint orbits, denoted $\mathfrak{O}^{L\mathfrak{g}}_\lambda$, of $\lambda+\rho$, ρ being the highest root. The kernels for each element in $\mathfrak{O}^{L\mathfrak{g}}_\lambda$ are given by the image under the co-adjoint action of the lowest-weight space of $\mathcal{H}_\lambda\otimes\mathcal{S}$, and give rise to a twisted, equivariant vector bundle localized on $\mathfrak{O}^{L\mathfrak{g}}_\lambda$.

From this data one can now construct a twisted equivariant K-theory class on G by using (4.6): namely, the co-adjoint action, say for the level 1 hyperplane $\Lambda + L\mathfrak{g}^*\subset\widehat{L\mathfrak{g}}$ is precisely given by the gauge action on a connection

$$g\cdot(\mu,1) = (\mathrm{ad}_g(\mu) - dgg^{-1}, 1)\,, \tag{4.7}$$

for $g\in LG$. Further, one can map a connection in $\mathfrak{A}$ to an element in G by utilizing the holonomy map

$$\mathrm{Hol}:\quad \mathfrak{A}\to G\,. \tag{4.8}$$

Then the co-adjoint action, a.k.a. gauge action, in (4.7) maps to

$$\mathrm{Hol}(g\cdot(\mu,1)) = g(0)\mathrm{Hol}(\mu)g(0)^{-1}\,. \tag{4.9}$$

In this way we can map the Dirac family on Lg^* to one on G, and the equivariance with respect to LG maps to an equivariance with respect to the adjoint action of G on itself. In summary, this construction thus gives rise to a twisted K-theory class on G, equivariant under the adjoint action of G.

4.2. Dirac-family construction for the chiral ring

Some related discussions for $H = T$ the maximal torus of G has appeared in [3], relating ${}^\tau K_G(G)$ to ${}^\tau K_T(T)$. For our purposes, we continue to assume only that G is simply-connected and G/H is hermitian symmetric, thus ensuring the existence of $\mathcal{N} = 2$ supersymmetry. Consider the affine Dirac operator $\not{\mathcal{D}} \equiv \not{\mathcal{D}}_{L\mathfrak{g}/L\mathfrak{h}} = \not{\mathcal{D}}^- + \not{\mathcal{D}}^+$ (*cf.* (4.3)) acting on $\mathcal{H}_\lambda \otimes \mathcal{S}_{L\mathfrak{p}}$ for $\mathcal{H}_\lambda$ an integrable highest weight representation of $\widehat{L\mathfrak{g}}_k$. This representation can be decomposed with respect to the subalgebra $\widehat{L\mathfrak{h}}$

$$\mathcal{H}_\lambda = \bigoplus_{\nu \in L\mathfrak{h}^*} M_{\lambda\nu} \mathcal{V}_\nu \,, \tag{4.10}$$

where $\mathcal{V}_\nu$ is a highest weight representation of $\widehat{L\mathfrak{h}}$, $M_{\lambda\nu}$ being the state spaces of the coset theory. Note that for the case of interest when G/H is hermitian symmetric, the Dirac operator simplifies drastically, as $\mathfrak{p}$ is commutative, so that $\not{\mathcal{D}} = J \cdot \psi$.

In [35] and [45] the kernel of $\not{\mathcal{D}}$ acting on $\mathcal{H}_\lambda \otimes \mathcal{S}_{L\mathfrak{p}}$ is determined as

$$\ker \not{\mathcal{D}}_{L\mathfrak{g}/L\mathfrak{h}} \big|_{\mathcal{H}_\lambda \otimes \mathcal{S}_{L\mathfrak{p}}} = \bigoplus_{(\lambda,\nu) \in \mathfrak{C}} \mathcal{V}_\nu \otimes \mathcal{S}_{L\mathfrak{p}} \,, \tag{4.11}$$

where[♯]

$$\mathfrak{C} = \Big\{ (\mu,\nu) \in L\mathfrak{g}^* \oplus L\mathfrak{h}^* \,;\; \nu + \hat\rho_{\mathfrak{h}} = \sigma(\mu + \hat\rho_{\mathfrak{g}}) \,, \quad \text{for some } \sigma \in \hat W_{\mathfrak{g}} / \hat W_{\mathfrak{h}} \Big\} \,. \tag{4.12}$$

The ring of chiral primaries $\mathcal{R}_{cp}^{(G,H)}$ contains strictly speaking only a subset of the chiral primaries in the kernel of the Dirac operator. More precisely, it is generated by the primaries in $\ker(\not{\mathcal{D}})$, which by (4.11) is identified with the elements of $\mathfrak{C}$, modulo field identifications, which reside in the common centre Z of G and H, *i.e.*,

$$\mathcal{R}_{cp} = \big\langle \Phi_{(\mu,\nu)} \,;\; (\mu,\nu) \in \mathfrak{C} \big\rangle \big/ Z \,. \tag{4.13}$$

♯ This is in fact the condition for RR groundstates, which we however shall use interchangeably with chiral primaries, via spectral flow. More precisely the set of chiral primaries is (μ,ν) with $\sigma(\mu + \rho_{\mathfrak{g}}) = \nu + \rho_{\mathfrak{g}}$. So we should denote by $\mathfrak{C}$ the set of coset weights satisfying this property.

The following construction will show that for each element in $\mathcal{R}_{cp}$ there exists a twisted H-equivariant K-theory class on H. Consider to begin with the complex defined by $\not{D}_{L\mathfrak{g}/L\mathfrak{h}}$ acting on $\mathcal{H}_\lambda \otimes \mathcal{S}_{L\mathfrak{p}}$. Define a family of Dirac operators on $L\mathfrak{h}^*$ as

$$\begin{aligned} L\mathfrak{h}^* \quad &\rightarrow \quad \mathrm{End}((\mathcal{H}_\lambda \otimes \mathcal{S}_\mathfrak{p}, \not{D}_{L\mathfrak{g}/L\mathfrak{h}})) \\ \mu \quad &\mapsto \quad \not{D}^\mu_{L\mathfrak{h}}\,, \end{aligned} \tag{4.14}$$

where we defined the (gauge-coupled) family of Dirac operators

$$\not{D}^\mu_{L\mathfrak{h}} = \not{D}_{L\mathfrak{h}} + i\psi(\mu)\,, \qquad \mu \in L\mathfrak{h}^*\,. \tag{4.15}$$

Taking the kernel of $\not{D}^\mu_{L\mathfrak{h}}$ on the complex $(\mathcal{H}_\lambda \otimes \mathcal{S}_\mathfrak{p}, \not{D}_{L\mathfrak{g}/L\mathfrak{h}})$ amounts to decomposing the kernel of $\not{D}_{L\mathfrak{g}/L\mathfrak{h}}$ as acting on $\mathcal{H}_\lambda \otimes \mathcal{S}$ into $\widehat{L\mathfrak{h}}$ representations, meaning the decomposition (4.11). So, this will result in a map from the Verlinde ring of ${}^\tau K_G(G)$ to the one of H, *i.e.*,

$$\begin{aligned} {}^\tau K_G(G) &\rightarrow {}^{\tau'} K_H(H) \\ \Phi^{\widehat{L\mathfrak{g}}}_\lambda &\mapsto \sum_{(\lambda,\nu)\in\mathfrak{C}} \Phi^{\widehat{L\mathfrak{h}}}_\nu\,. \end{aligned} \tag{4.16}$$

Note that the elements of the chiral ring can also be viewed as semi-infinite Lie algebra cohomology elements, as discussed in [35]. Decomposing $L\mathfrak{p} = L\mathfrak{p}^+ \oplus L\mathfrak{p}^-$ such that $[\mathfrak{p}^\pm, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$, the kernel of $\not{D}_{L\mathfrak{g}/L\mathfrak{h}}$ acting on $\mathcal{H}_\lambda \otimes \mathcal{S}_{L\mathfrak{p}}$ is

$$\ker \not{D}_{L\mathfrak{g}/L\mathfrak{h}}\big|_{\mathcal{H}_\lambda\otimes\mathcal{S}_{L\mathfrak{p}}} = H^*(L\mathfrak{p}^*; \mathcal{H}_\lambda)\,. \tag{4.17}$$

In this way one obtains a map from $V_k(G) \rightarrow V_{k+h^\vee_\mathfrak{g}-h^\vee_\mathfrak{h}}(H)$, mapping precisely as in (4.16).

5. Discussions and Outlook

In summary, we have defined a boundary ring $\mathcal{B}_k$ for $\mathcal{N} = 2$ coset models (G, H) in terms of the twisted equivariant K-theory ${}^\tau K_{H/Z}(G)$. The rank of $\mathcal{B}_k$ agrees with that of the chiral ring of the coset model, however the product structures differ. Both rings are quotients of the representation ring of H, with respect to the Verlinde ideals of G and of H, at specific levels, respectively.

The present analysis is somewhat reminiscent of the theorem by Freed-Hopkins-Teleman, which identifies the Verlinde algebra $V_k(G)$ with the twisted equivariant K-theory ${}^\tau K_G(G)$. Naively, one might have anticipated an isomorphism between ${}^\tau K_{H/Z}(G)$ and the

chiral ring, which not only respects the structure as abelian groups, but also the *products*. This is however not the case. More to the point, the present result suggests that the K-theory associated to a particular sigma-model gives rise to an algebra on classes of D-branes. This may well tie in with the algebra of BPS states defined by Harvey and Moore [5,6]. The natural interpretation of FHT in this light is, that for the topological G/G coset model the D-brane charge relations obey the Verlinde algebra of G.

It would be interesting to understand, what the precise relation between $\mathcal{B}_k$ and the chiral ring is, *e.g.*, if one can be obtained as a deformation of the other, and possible relations to the quantum cohomology ring may be interesting to explore. The boundary ring in the case of superminimal models, discussed in section 3.2, turned out to be a deformation of the chiral ring (by taking essentially only the highest degree component of the fusion ideal). In this case, the boundary ring is a quantum deformed version of the bulk chiral ring. Two immediate questions arise: firstly, whether this holds for all KS coset models, *i.e.*, the boundary ring is given by a quantum (or otherwise) deformed bulk chiral ring and secondly, whether this has any implications upon twisted K-theory.

One spin-off of our results is the agreement of the rank of the charge lattice of the D-branes with the rank of the chiral ring. This is a result that had been anticipated already in [4] and suggests that the Cardy boundary states provide a complete basis for the charge lattice. One can in fact define a product on these Cardy boundary states (and this will be discussed in detail in [7]), which should then agree with the product in the boundary ring $\mathcal{B}_k$. Another interesting line of thought would be to study the D-branes in the topologically twisted KS models in this light. Such a worldsheet derivation of the boundary ring, whether in the full CFT or in the topologically twisted model, will certainly substantiate the proposal put forward in this paper, and may help elucidating the relation between boundary conformal field theory and K-theory.

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Appendix A. Computation of the ranks of the Verlinde algebras

In this appendix we shall prove explicit formulae for the rank of the fusion ring $d_k(G)$ for the groups G appearing in table 1. For the groups in question, the relations, of which for fixed k one needs to enumerate the non-negative integral solutions for $\{\Lambda^{(i)}\}$, are

$$\begin{aligned}
A_n = SU(n+1) : &\qquad \sum_{i=1}^{n} \Lambda^{(i)} \leq k \\
D_n = SO(2n) : &\qquad \Lambda^{(1)} + \Lambda^{(n-1)} + \Lambda^{(n)} + 2\sum_{i=2}^{n-2} \Lambda^{(i)} \leq k \\
B_n = SO(2n+1) : &\qquad \Lambda^{(1)} + \Lambda^{(n)} + 2\sum_{i=2}^{n-1} \Lambda^{(i)} \leq k \\
C_n = Sp(2n) : &\qquad \sum_{i=1}^{n} \Lambda^{(i)} \leq k \\
E_6 : &\qquad \Lambda^{(1)} + \Lambda^{(5)} + 2(\Lambda^{(2)} + \Lambda^{(4)} + \Lambda^{(6)}) + 3\Lambda^{(3)} \leq k \\
E_7 : &\qquad \Lambda^{(6)} + 2(\Lambda^{(1)} + \Lambda^{(5)} + \Lambda^{(7)}) + 3(\Lambda^{(2)} + \Lambda^{(4)}) + 4\Lambda^{(3)} \leq k\,.
\end{aligned} \tag{A.1}$$

In summary we obtain table 2, where k is the level of the affine algebra corresponding to G, $\kappa \in \mathbb{N}_0$ and $n \in \mathbb{N}$.

G	k	$d_k(G)$
A_n	k	$\binom{n+k}{k}$
B_{n+2}	2κ	$\binom{n+\kappa+1}{n+1} + 4\binom{n+\kappa+1}{n+2}$
B_{n+2}	$2\kappa+1$	$3\binom{n+\kappa+1}{n+1} + 4\binom{n+\kappa+1}{n+2}$
C_n	k	$\binom{n+k}{k}$
D_{n+3}	2κ	$\binom{n+\kappa+1}{n+1} + 8\binom{n+\kappa+2}{n+3}$
D_{n+3}	$2\kappa+1$	$4\binom{n+\kappa+2}{n+2} + 8\binom{n+\kappa+2}{n+3}$
E_6	k	(A.14)
E_7	k	(A.17)

Table 2: *Ranks $d_k(G)$ of the Verlinde algebras $V_k(G)$.*

The remainder of this appendix will give the derivations of the relations in table 2. For $SU(n+1)$ and $Sp(2n)$ the argument is straight forward. To prove the assertion in this case one can proceed by induction upon k. The case $k=0$ is trivially satisfied. The induction step follows by using

$$d_{k-1}(G) + \binom{n+k-1}{k} = \binom{n+k-1}{k-1} + \binom{n+k-1}{k} = \binom{n+k}{k} = d_k(G)\,, \qquad \text{(A.2)}$$

which implies the required formula, as the number of partitions of k into n parts is $\binom{n+k-1}{k}$.

Next, we consider $G = B_n = SO(2n+1)$. By using the result of $SU(2n+1)$ the rank of the fusion ring for $SO(2n+1)$ $(n \geq 3)$ is

$$d_k(SO(2n+1)) = \sum_{\Lambda^{(1)}=0}^{k} \sum_{\Lambda^{(n)}=0}^{k-\Lambda^{(1)}} \binom{n-2+\left[\frac{k-\Lambda^{(1)}-\Lambda^{(n)}}{2}\right]}{n-2} = \sum_{\Lambda=0}^{k} (\Lambda+1) \binom{n-2+\left[\frac{k-\Lambda}{2}\right]}{n-2}\,. \qquad \text{(A.3)}$$

Using

$$\sum_{l=0}^{k} \binom{n+l}{n} = \binom{n+1+k}{n+1}\,, \qquad \sum_{l=0}^{k} l \binom{n+l}{n} = (n+1)\binom{n+k+1}{n+2}\,, \qquad \text{(A.4)}$$

one can sum these to obtain for even level $2k$

$$\begin{aligned} & d_{2k}(SO(2(N+2)+1)) \\ &= -2(N+1)\binom{N+k+1}{N+2} + (2k+1)\binom{N+k+1}{N+1} - 2(N+1)\binom{N+k}{N+2} + 2k\binom{N+k}{N+1} \\ &= -2N\binom{N+k+1}{N+2} + (2k+1)\binom{N+k+1}{N+1}\,. \end{aligned} \qquad \text{(A.5)}$$

This formula can be further simplified straight-forwardly by *e.g.*, expanding out one of the binomial coefficients

$$d_{2k}(SO(2(N+2)+1)) = \binom{N+k+1}{N+1} + 4\binom{N+k+1}{N+2}\,. \qquad \text{(A.6)}$$

For odd level $2k+1$ the rank is computed by

$$\begin{aligned} d_{2k+1}(SO(2(N+2)+1)) &= (4\kappa+3)\binom{N+\kappa+1}{N+1} - 4(N+1)\binom{N+\kappa+1}{N+2} \\ &= 3\binom{N+\kappa+1}{N+1} + 4\binom{N+\kappa+1}{N+2}\,. \end{aligned} \qquad \text{(A.7)}$$

Next consider D_n. We proceed analogously by computing

$$\begin{aligned} d_k(SO(2n)) &= \sum_{\Lambda^{(1)}=0}^{k} \sum_{\Lambda^{(n)}=0}^{k-\Lambda^{(1)}} \sum_{\Lambda^{(n-1)}=0}^{k-\Lambda^{(1)}-\Lambda^{(n)}} \binom{n-3+\left[\frac{k-\Lambda^{(1)}-\Lambda^{(n)}-\Lambda^{(n-1)}}{2}\right]}{n-3} \\ &= \sum_{\Lambda=0}^{k} \frac{(\Lambda+1)(\Lambda+2)}{2} \binom{n-3+\left[\frac{k-\Lambda}{2}\right]}{n-3} . \end{aligned} \tag{A.8}$$

In order to evaluate this, note that

$$\begin{aligned} \sum_{l=0}^{k} l^2 \binom{n+l}{n} &= (n+1)(n+2)\binom{n+k+2}{n+3} - (n+1)^2 \binom{n+k+1}{n+2} \\ &= (n+1)(n+2)\binom{n+k+1}{n+3} + (n+1)\binom{n+k+1}{n+2} . \end{aligned} \tag{A.9}$$

Evaluation of the sum (A.8) yields for even level $k = 2\kappa$

$$\begin{aligned} d_{2\kappa}(D_{n+3}) &= \sum_{\Lambda=0}^{k} (2\kappa-2\Lambda+1)^2 \binom{n+\Lambda}{n} \\ &= (2\kappa+1)^2 \binom{n+\kappa+1}{n+1} - 4(n+1)\binom{n+\kappa+2}{n+3} - 4\kappa(n+1)\binom{n+\kappa+1}{n+2} \\ &= \binom{n+\kappa+1}{n+1} + 8\binom{n+\kappa+2}{n+3} . \end{aligned} \tag{A.10}$$

For odd level the rank is computed to be

$$\begin{aligned} d_{2\kappa+1}(D_{n+3}) &= \sum_{\Lambda=0}^{k} 4(\kappa-\Lambda+1)^2 \binom{n+\Lambda}{n} \\ &= 4(\kappa+1)^2 \binom{n+\kappa+1}{n+1} - 4(n+1)\binom{n+\kappa+2}{n+3} - 4(\kappa+1)(n+1)\binom{n+\kappa+1}{n+2} \\ &= 4\binom{n+\kappa+2}{n+2} + 8\binom{n+\kappa+2}{n+3} . \end{aligned} \tag{A.11}$$

The case of E_6 can be derived by using the result for B_5

$$d_k(E_6) = \sum_{\Lambda=0}^{k} \left(\left[\frac{k-\Lambda}{3}\right] + 1 \right) \left(d_\Lambda(B_5) - d_{\Lambda-1}(B_5)\right) . \tag{A.12}$$

There are three cases to be considered: $k = 3\kappa$, $k = 3\kappa+1$ and $k = 3\kappa+2$, for $\kappa \in \mathbb{N}_0$. In each of these cases we obtain the following sums

$$d_{3\kappa+s}(E_6) = \sum_{\Lambda=0}^{\kappa} (\kappa-\Lambda+1) \left(d_{3\Lambda+s}(B_5) - d_{3\Lambda-3+s}(B_5)\right) \tag{A.13}$$

$$= \left(\sum_{\Lambda=0}^{\kappa} d_{3\Lambda+s}(B_5) \right) - (\kappa+1) d_{s-3}(B_5) \ , \qquad s = 0,1,2\,. \tag{A.14}$$

Note, that $d_\kappa(G) = 0$ for $\kappa < 0$.

For E_7 one again proceeds stepwise. For fixed value of $k_4 = \Lambda^{(6)} + 2(\Lambda^{(1)} + \Lambda^{(5)} + \Lambda^{(7)})$ one has

$$\sum_{\Lambda^{(6)}=0}^{k_4} \binom{3 + \left[\frac{k_4 - \Lambda^{(6)}}{2}\right]}{3} , \tag{A.15}$$

and for fixed $k_6 = \Lambda^{(6)} + 2(\Lambda^{(1)} + \Lambda^{(5)} + \Lambda^{(7)}) + 3(\Lambda^{(2)} + +\Lambda^{(4)})$

$$n_{k_6} = \sum_{\Lambda^{(4)}=0}^{k_6} \sum_{\Lambda^{(2)}=0}^{k_6 - \Lambda^{(4)}} \sum_{\Lambda^{(6)}=0}^{\left[\frac{k_6 - \Lambda^{(2)} - \Lambda^{(4)}}{3}\right]} \binom{3 + \left[\frac{\left[\frac{k_6 - \Lambda^{(2)} - \Lambda^{(4)}}{3}\right] - \Lambda^{(6)}}{2}\right]}{3} \tag{A.16}$$

Iterating this procedure, we arrive at

$$d_k(E_7) = \sum_{\Lambda^{(3)}=0}^{k} n_{\Lambda^{(3)}} \ . \tag{A.17}$$

In summary we obtain table 2.

Note that these multiplicity formulae can also be extracted from the generating function

$$\begin{aligned} A_{(a_1,\cdots,a_n)}(q) &= (1-q)^{-1} \prod_{i=1}^{n} (1 - q^{a_i})^{-1} \\ &= \sum_{k=0}^{\infty} d_k(G) q^k \,, \end{aligned} \tag{A.18}$$

which counts the non-negative integer solutions $\{\Lambda^{(i)}\}$ to

$$a_1 \Lambda^{(1)} + \cdots + a_n \Lambda^{(n)} \leq k \,, \tag{A.19}$$

where the choice of group G determines the coefficients $a_i \in \mathbb{N}_0$ as *e.g.* in (A.1).

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